

Relevance and interpolation via binary nested sequents

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Joint work with Fabio De Martin Polo (IUSS, Pavia)

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Relevance and Craig interpolation in relevant logic

Relevance properties.

- **Variable sharing** (VSP):

if $\vdash A \rightarrow B$ then $\text{Var}(A) \cap \text{Var}(B) \neq \emptyset$.

- **Depth relevance** (DRP): some shared variable occurs at the same $\rightarrow$ -depth in A and B .

Both are usually proved *semantically*.

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- RM^t and some extensions have the CIP [Marchioni and Metcalfe, 2012] $\Rightarrow$ **RM^t lacks VSP**
- it is not currently known whether any relevant logic between B and R enjoys CIP [Standfer, 2026]
- the base logic B and its positive fragment B^+ : not settled [Knudstorp, 2026]

In this talk...

B^+ has the Craig interpolation property

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Along the way

1. A cut-free calculus by means of *binary nested sequents*
2. *syntactic* proofs of VSP, DRP and other relevance properties
3. *constructive Craig interpolants*, computed rule by rule from a restricted form of proof

The logic B^+

Language: $A ::= p \mid A \wedge A \mid A \vee A \mid A \rightarrow A$

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Routley–Meyer semantics

A frame $\langle K, O, R \rangle$ has set of worlds K , set of regular worlds $O \subseteq K$ and a ternary R . Put $a \leq b$ iff $Rxab$ for some $x \in O$, and require

(F1) $a \leq a$;

(F2) $a \leq b$ and $Rbcd$ imply $Racd$.

Valuations are upward closed along $\leq$, and

$$a \Vdash A \rightarrow B \iff \forall b, c (Rabc \ \& \ b \Vdash A \Rightarrow c \Vdash B)$$

A is valid if it holds at every regular world of every model.

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Hilbert system $\mathcal{H}_{B^+}$

ax1 $A \rightarrow A$

ax2 $A_1 \wedge A_2 \rightarrow A_i$

ax3 $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$

ax4 $A_i \rightarrow A_1 \vee A_2$

ax5 $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$

ax6 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

mp from A and $A \rightarrow B$ infer B

adj from A and B infer $A \wedge B$

aff from $A \rightarrow B$ and $C \rightarrow D$ infer
 $(B \rightarrow C) \rightarrow (A \rightarrow D)$

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Theorem ([Routley et al., 1982])

A is derivable in $\mathcal{H}_{B^+}$ if and only if A is B^+ -valid.

Binary nested sequents

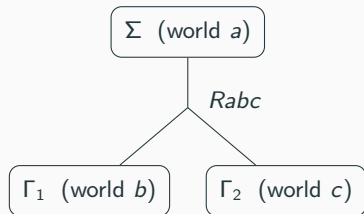
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Recall the Routley–Meyer clause for $\rightarrow$:

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In the syntax, the nesting is **binary**:

$\Sigma, \langle \Gamma_1 \parallel \Gamma_2 \rangle$ represents a with the two successors b, c



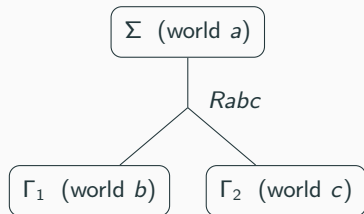
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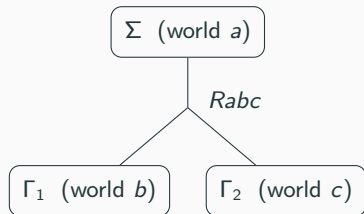
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Grammar:

$\Gamma :: \Lambda, \Pi$ (full) $\Lambda :: \emptyset \mid A^\bullet \mid \langle \Lambda // \Lambda \rangle \mid \Lambda, \Lambda$ (l-sided) $\Pi :: A^\circ \mid \langle \Lambda // \Gamma \rangle \mid \langle \Gamma // \Lambda \rangle$ (r-sided)

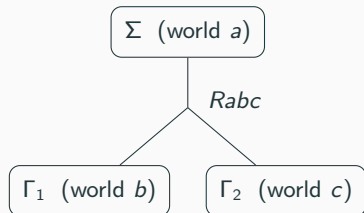
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Example

$\Sigma = p \vee q^\bullet, \langle p^\bullet // p \vee q^\bullet \rangle, \langle q^\circ // p \rightarrow q^\bullet \rangle$ is a full nested sequent,

$\Lambda = p \vee q^\bullet, \langle p^\bullet // p \vee q^\bullet \rangle, \langle p^\bullet, q^\bullet // p \rightarrow q^\bullet \rangle$ is a left-sided sequent.

$$\frac{G\{\Sigma, \langle A^\bullet // B^\circ \rangle\}}{G\{\Sigma, A \rightarrow B^\circ\}} \rightarrow^\circ$$

$$\frac{G^*\{\Sigma^*, A \rightarrow B^\bullet, \langle \Phi_1^*, A^\circ // \Phi_2^* \rangle\} \quad G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 // \Phi_2, B^\bullet \rangle\}}{G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 // \Phi_2 \rangle\}} \rightarrow^\bullet$$

- $G\{ \}$ is a placeholder called context, emptying one relevant component
- $(\cdot)^*$ deletes the unique output formula
- $\rightarrow^\circ$ creates a block $\langle A^\bullet // B^\circ \rangle$
- $\rightarrow^\bullet$ propagates antecedent A to the first component and consequent B to the second

One **primitive structural rule**, non-nested, i.e. applicable only to top-level blocks:

$$\frac{\Sigma, \langle A^\bullet, \Phi_1 // A^\bullet, \Phi_2 \rangle}{\Sigma, \langle A^\bullet, \Phi_1 // \Phi_2 \rangle} \text{ (lift)}$$

which expresses monotonicity along $\leq$ (regular worlds)

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The following rules are **height-preserving admissible**:

$$\frac{G\{\Sigma\}}{G\{\Lambda, \Sigma\}} \text{ w} \quad \frac{G\{A^\bullet, A^\bullet, \Sigma\}}{G\{A^\bullet, \Sigma\}} \text{ c} \quad \frac{G\{\Sigma, \langle \Phi_1 // \Phi_2 \rangle, \langle \Psi_1 // \Psi_2 \rangle\}}{G\{\Sigma, \langle \Phi_1, \Psi_1 // \Psi_2, \Phi_2 \rangle\}} \text{ m}$$

Structural rules

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where R1 reflects (F1): $a \leq a$ R2 reflects (F2): $a \leq b$ and $Rbcd$ imply $Racd$

The nested calculus C_{B+}

Initial sequents:

$$\frac{}{G\{\Sigma, p^\bullet, p^\circ\}} \text{ (id)}$$

Logical rules:

$$\frac{G\{\Sigma, A^\bullet, B^\bullet\}}{G\{\Sigma, A \wedge B^\bullet\}} \wedge^\bullet$$

$$\frac{G\{\Sigma, A^\circ\} \quad G\{\Sigma, B^\circ\}}{G\{\Sigma, A \wedge B^\circ\}} \wedge^\circ$$

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$$\frac{G^*\{\Sigma^*, A \rightarrow B^\bullet, \langle \Phi_1^*, A^\circ // \Phi_2^* \rangle\} \quad G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 // \Phi_2, B^\bullet \rangle\}}{G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 // \Phi_2 \rangle\}} \rightarrow^\bullet$$

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Therefore,

$$\vdash_{\mathcal{H}_{B^+}} F \iff \vdash_{\mathbf{C}_{B^+} + (\text{cut})} F^\circ \iff \vdash_{\mathbf{C}_{B^+}} F^\circ$$

Consequently, for any Φ (block-free) and Ψ ,

$$\vdash_{\mathbf{C}_{B^+}} \langle \Phi // \Psi \rangle \iff \vdash_{\mathbf{C}_{B^+}} \langle - // \Phi, \Psi \rangle \text{ called } \mathbf{0}\text{-free sequent}$$

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We will only consider derivations of 0-free sequents for deriving the meta-logical properties

A running example

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Var(A) p, q, r, u, v

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Two natural guesses for the interpolant

Erase u, v from A : $(p \rightarrow r) \wedge (q \rightarrow r)$

Erase s, t from B : $(p \vee q) \rightarrow r$

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Plan

1. derive $\langle - // A^\bullet, B^\circ \rangle$ in the calculus
2. read off *variable sharing* and *depth relevance* from a proof
3. extract an interpolant θ with $\text{Var}(\theta) \subseteq \{p, q, r\}$ and both $A \rightarrow \theta$ and $\theta \rightarrow B$ are provable

Derivation

$$\begin{array}{c}
 \frac{\mathcal{D}_1}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel r \vee t^\circ \rangle \rangle} \quad \frac{\mathcal{D}_2}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle q^\bullet, s^\bullet \parallel r \vee t^\circ \rangle \rangle} \vee^\bullet \\
 \hline
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 \end{array}$$

Subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$

$\mathcal{D}_1$:

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 \frac{\frac{}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet, \textcolor{blue}{p}^\circ \parallel - \rangle \rangle} \text{id} \quad \frac{\frac{}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel r^\bullet, u^\bullet, \textcolor{red}{r}^\circ \rangle \rangle} \text{id} \quad \frac{}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel \textcolor{red}{r} \wedge \textcolor{red}{u}^\bullet, \textcolor{red}{r}^\circ \rangle \rangle} \wedge^\bullet}{\frac{}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel \textcolor{red}{r} \wedge \textcolor{red}{u}^\bullet, \textcolor{red}{r}^\circ \rangle \rangle} \rightarrow^\bullet} \\
 \frac{\frac{}{\langle - \parallel \textcolor{red}{p} \rightarrow (\textcolor{red}{r} \wedge \textcolor{red}{u})^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel \textcolor{red}{r}^\circ \rangle \rangle} \vee^\circ}{\langle - \parallel p \rightarrow (r \wedge u)^\bullet, q \rightarrow (r \wedge v)^\bullet, \langle p^\bullet, s^\bullet \parallel \textcolor{red}{r} \vee \textcolor{red}{t}^\circ \rangle \rangle} \vee^\circ
 \end{array}$$

$\mathcal{D}_2$:

$$\begin{array}{c}
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$$\Phi_1 = p^\bullet, \langle q^\bullet // p^\circ \rangle \quad \Phi_2 = t^\bullet \quad \Phi_3 = q^\bullet \quad \Phi_4 = p^\circ$$

we have $\widetilde{\Phi}_3$ is reachable to both $\widetilde{\Phi}_1$ and $\widetilde{\Sigma}$ but is not reachable to $\widetilde{\Phi}_4$ nor $\widetilde{\Phi}_2$.

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Let $P_1 := p \rightarrow (r \wedge u)$ and $Q_1 := q \rightarrow (r \wedge v)$, so $A = P_1 \wedge Q_1$ and take one branch in $\mathcal{D}$

rule	conclusion	output	witness $\rho^\bullet$	location
id	$\langle - \parallel P_1^\bullet, Q_1^\bullet, \langle p^\bullet, s^\bullet \parallel r^\bullet, u^\bullet, r^\circ \rangle \rangle$	r°	$r^\bullet$	same node
$\wedge^\bullet$	$\langle - \parallel P_1^\bullet, Q_1^\bullet, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle \rangle$	r°	$r \wedge u^\bullet$	same node
$\rightarrow^\bullet$	$\langle - \parallel P_1^\bullet, Q_1^\bullet, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle$	r°	$P_1^\bullet$	parent node
$\vee^\circ, \vee^\bullet, \wedge^\bullet$	...	$r \vee t^\circ$	$P_1^\bullet$	parent node
$\rightarrow^\circ$	$\langle - \parallel P_1^\bullet, Q_1^\bullet, B^\circ \rangle$	B°	$P_1^\bullet$	same node
$\wedge^\bullet$	$\langle - \parallel A^\bullet, B^\circ \rangle$	B°	$A^\bullet$	root

Lemma (VSP for sequents)

Let $\Gamma = \langle - // \Sigma \rangle$ be a 0-free nested sequent with the output formula θ . If it is provable, then there exist $\Phi, \Psi \in^+ \Sigma$ such that

- $\theta^\circ \in \tilde{\Phi}$;
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Proof.

By induction on the height h of the derivation of Γ . □

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rule	conclusion	output	witness $\rho^\bullet$	depth change
id	$\langle - \parallel P_1^\bullet, Q_1^\bullet, \langle p^\bullet, s^\bullet \parallel r^\bullet, u^\bullet, r^\circ \rangle \rangle$	r°	$r^\bullet$	-
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$\rightarrow^\bullet$	$\langle - \parallel P_1^\bullet, Q_1^\bullet, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle$	r°	$P_1^\bullet$	for $\hat{r}$ in P_1 , formula depth +1, sequent depth -1
$\vee^\circ, \vee^\bullet, \wedge^\bullet$	...	$r \vee t^\circ$	$P_1^\bullet$	-
$\rightarrow^\circ$	$\langle - \parallel P_1^\bullet, Q_1^\bullet, B^\circ \rangle$	B°	$P_1^\bullet$	for $\hat{r}$ in B , formula depth +1, sequent depth -1
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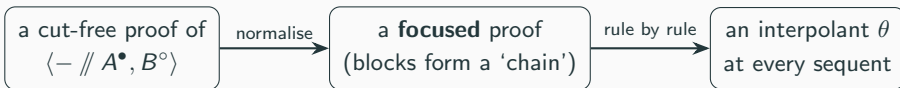
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Reachability	+	polarity	+	depth	$\Rightarrow$	proof of SDRP
Reachability	+			depth	$\Rightarrow$	proof of DRP
Reachability	+	polarity			$\Rightarrow$	proof of SVSP
Reachability					$\Rightarrow$	proof of VSP

A running example: CIP

Goal. From a proof of $\langle - \parallel A^\bullet, B^\circ \rangle$, compute θ with $\vdash A \rightarrow \theta$, $\vdash \theta \rightarrow B$, $\text{Var}(\theta) \subseteq \text{Var}(A) \cap \text{Var}(B)$.



Focused proofs and normal applications

A sequent Σ is called **focused** if for every nested block $\langle \Phi_1 // \Phi_2 \rangle$ in it, either Φ_1 or Φ_2 is full. We say a proof is **focused** if it contains focused sequents only.

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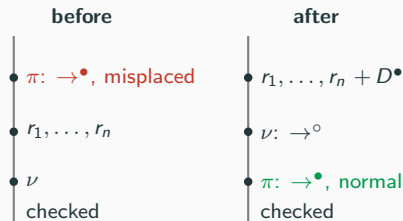
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Consequence

Every $\rightarrow$ -application contained in a focused proof of $\langle - // A^\bullet, B^\circ \rangle$ is normal.

Existence of focused proofs

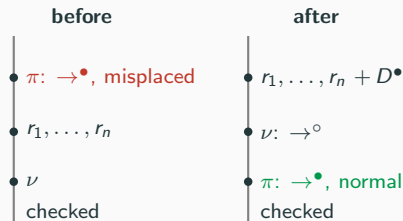


Check a given proof **from the root upwards**. At the first *misplaced* (non-normal) $\rightarrow^\bullet \pi$ on a branch, everything below is already focused, and:

output inside the **first** component: *delete* π ;

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Termination

Each step *shrinks* the part still to be checked, so the search terminates and returns a focused proof.

Constructing interpolation

We only use the innermost structure contained in a nested sequent to construct interpolants.

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	shape of active unit	$\mathcal{I}_L(\theta)$	$\mathcal{I}_R(\theta)$	V_L / V_R
(a)	$\Psi, \langle \Phi_1, F^\circ // \Phi_2 \rangle$	$\langle - // \Phi_1, \theta^\circ \rangle$	$\langle - // \theta^\bullet, F^\circ \rangle$	$\text{Var}(\Phi_1) / \text{Var}(F)$
(b)	$\Psi, \langle \Phi_1 // \Phi_2, F^\circ \rangle, \Phi_2 \neq \emptyset$	$\langle - // \Psi, \langle \Phi_1 // \Phi_2, \theta^\circ \rangle \rangle$	$\langle - // \theta^\bullet, F^\circ \rangle$	$\text{Var}(\Psi, \Phi_2) / \text{Var}(F)$
(c)	$\Psi, \langle \Phi_1 // F^\circ \rangle$	$\langle - // \Psi, \theta^\circ \rangle$	$\langle - // \theta^\bullet, \langle \Phi_1 // F^\circ \rangle \rangle$	$\text{Var}(\Psi) / \text{Var}(\Phi_1, F)$

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Invariant. Both $\mathcal{I}_L(\theta)$ and $\mathcal{I}_R(\theta)$ are provable, and $\text{Var}(\theta) \subseteq V_L \cap V_R$.

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At the root

$\langle - \parallel A^\bullet, B^\circ \rangle$ is case (b) with $\Psi = \Phi_1 = \emptyset$, $\Phi_2 = \{A\}$, $F = B$.

By $\mathcal{I}_L(\theta)$, $\langle - \parallel \langle - \parallel A^\bullet, \theta^\circ \rangle \rangle$ is provable, by R1, so is $\langle - \parallel A^\bullet, \theta^\circ \rangle$, hence $A \rightarrow \theta$ is provable;

By $\mathcal{I}_R(\theta)$, $\langle - \parallel B^\circ, \theta^\bullet \rangle$ is provable. hence $\theta \rightarrow B$ is provable.

The running example

The branch $\mathcal{D}_1$, with $\Pi := P_1^\bullet, Q_1^\bullet$; each sequent is marked with its case.

$$\begin{array}{c}
 \frac{}{S_\ell : \langle - // \Pi, \langle p^\bullet, s^\bullet, p^\circ // - \rangle \rangle \text{ (a)}} \text{id} \quad \frac{\frac{}{\langle - // \Pi, \langle p^\bullet, s^\bullet // r^\bullet, u^\bullet, r^\circ \rangle \rangle} \text{id}}{\frac{}{\langle - // \Pi, \langle p^\bullet, s^\bullet // r \wedge u^\bullet, r^\circ \rangle \rangle \text{ (b)}}} \wedge^\bullet \\
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 \frac{S_\ell : \langle - // \Pi, \langle p^\bullet, s^\bullet, p^\circ // - \rangle \rangle \text{ (a)} \quad \frac{}{S_r : \langle - // \Pi, \langle p^\bullet, s^\bullet // r \wedge u^\bullet, r^\circ \rangle \rangle \text{ (b)}} \wedge^\bullet}{\frac{}{\rightarrow^\bullet \text{ on } P_1}} \\
 \hline
 \frac{S_5 : \langle - // \Pi, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle \text{ (c)}}{S_4 : \langle - // \Pi, \langle p^\bullet, s^\bullet // r \vee t^\circ \rangle \rangle \text{ (c)}} \vee^\circ \quad \frac{\mathcal{D}_2}{S'_4 \text{ (c)}} \\
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 \frac{S_3 : \langle - // \Pi, \langle (p \vee q)^\bullet, s^\bullet // r \vee t^\circ \rangle \rangle \text{ (c)}}{S_2 : \langle - // \Pi, \langle ((p \vee q) \wedge s)^\bullet // r \vee t^\circ \rangle \rangle \text{ (c)}} \wedge^\bullet \\
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 \hline
 \frac{S_1 : \langle - // \Pi, B^\circ \rangle \text{ (b)}}{S_0 : \langle - // A^\bullet, B^\circ \rangle \text{ (b)}} \wedge^\bullet
 \end{array}$$

Construction I: basic cases

Left premiss S_ℓ

$$S_\ell = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet, p^\circ \parallel - \rangle \rangle$$

$$\text{active unit (case (a))} = \Pi, \langle p^\bullet, s^\bullet, p^\circ \parallel - \rangle$$

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Next, S_r

$$S_r = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle \rangle;$$

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$$\mathcal{I}_R(\theta) = \langle - \parallel p^\bullet, p^\circ \rangle$$

$$V_L = \{p, s\}, \quad V_R = \{p\}$$

Right premiss similar to S_ℓ , $\theta = r$;

Next, S_r

$$S_r = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle \rangle;$$

$$\text{active unit (case (b))} = \Pi, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle$$

$$\theta_r := \theta = r$$

$$\mathcal{I}_L(\theta) = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle \rangle$$

$$\mathcal{I}_R(\theta) = \langle - \parallel r^\bullet, r^\circ \rangle$$

$$V_L = \{p, q, r, u, v\}, \quad V_R = \{r\}$$

Construction II: the case of $\rightarrow^\bullet$

$S_5 = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle$ is derived by $\rightarrow^\bullet$:

$$\frac{\frac{S_\ell : \langle - \parallel \Pi, \langle p^\bullet, s^\bullet, p^\circ \parallel - \rangle \rangle \text{ (a)}}{\quad} \quad \frac{S_r : \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r \wedge u^\bullet, r^\circ \rangle \rangle \text{ (b)}}{\quad}}{S_5 : \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle \text{ (c)}} \rightarrow^\bullet \text{ on } P_1$$

Construction II: the case of $\rightarrow^\bullet$

$S_5 = \langle - \parallel \Pi, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle$ is derived by $\rightarrow^\bullet$:

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active unit in case (c): $\mathcal{I}_L(\theta) = \langle - \parallel \Pi, \theta^\circ \rangle \quad \mathcal{I}_R(\theta) = \langle - \parallel \theta^\bullet, \langle p^\bullet, s^\bullet \parallel r^\circ \rangle \rangle$

Construction II: the case of $\rightarrow^\bullet$

$S_5 = \langle - // \Pi, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$ is derived by $\rightarrow^\bullet$:

$$\frac{\overline{S_\ell : \langle - // \Pi, \langle p^\bullet, s^\bullet, p^\circ // - \rangle \rangle} \quad (a) \quad \overline{S_r : \langle - // \Pi, \langle p^\bullet, s^\bullet // r \wedge u^\bullet, r^\circ \rangle \rangle} \quad (b)}{S_5 : \langle - // \Pi, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle} \rightarrow^\bullet \text{ on } P_1 \quad (c)$$

active unit in case (c): $\mathcal{I}_L(\theta) = \langle - // \Pi, \theta^\circ \rangle$ $\mathcal{I}_R(\theta) = \langle - // \theta^\bullet, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$

from S_ℓ :

$$\theta_\ell = p$$



from S_r :

$$\theta_r = r$$



$$\theta = \theta_\ell \rightarrow \theta_r = p \rightarrow r$$

- $\mathcal{I}_L(\theta) = \langle - // \Pi, p \rightarrow r^\circ \rangle$: $\rightarrow^\circ$, then $\rightarrow^\bullet$ on $P_1 = p \rightarrow (r \wedge u)$
- $\mathcal{I}_R(\theta) = \langle - // p \rightarrow r^\bullet, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$: $\rightarrow^\bullet$ on $p \rightarrow r$

Construction II: the case of $\rightarrow^\bullet$

$S_5 = \langle - // \Pi, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$ is derived by $\rightarrow^\bullet$:

$$\frac{\overline{S_\ell : \langle - // \Pi, \langle p^\bullet, s^\bullet, p^\circ // - \rangle \rangle} \quad (a) \quad \overline{S_r : \langle - // \Pi, \langle p^\bullet, s^\bullet // r \wedge u^\bullet, r^\circ \rangle \rangle} \quad (b)}{S_5 : \langle - // \Pi, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle} \rightarrow^\bullet \text{ on } P_1 \quad (c)$$

active unit in case (c): $\mathcal{I}_L(\theta) = \langle - // \Pi, \theta^\circ \rangle$ $\mathcal{I}_R(\theta) = \langle - // \theta^\bullet, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$

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- $\mathcal{I}_R(\theta) = \langle - // p \rightarrow r^\bullet, \langle p^\bullet, s^\bullet // r^\circ \rangle \rangle$: $\rightarrow^\bullet$ on $p \rightarrow r$

Remark

The step $\theta := \theta_\ell \rightarrow \theta_r$ does not always work. The $\mathcal{I}_L$ -part needs θ_ℓ to *replace* the whole first component inside the right premiss, which can fail. In this case we replace $p^\bullet, s^\bullet$ with $\theta_\ell = p^\bullet$. The formal construction writes $\alpha := \theta_\ell$ and strengthens it to some α' with $\Phi_1 \vdash \alpha' \vdash \alpha$; here $\alpha' = \alpha = \theta_\ell$.

	case	rule	θ
S_5	(c)	$\rightarrow^\bullet, \Phi_2 = \emptyset$	$p \rightarrow r$
S_4	(c)	$\vee^\circ$ on F	$p \rightarrow r$
S'_4	(c)	—	$q \rightarrow r$
S_3	(c)	$\vee^\bullet$ in Φ_1	$(p \rightarrow r) \wedge (q \rightarrow r)$
S_2	(c)	$\wedge^\bullet$ in Φ_1	same
S_1	(b)	$\rightarrow^\circ$	same
S_0	(b)	$\wedge^\bullet$ in Φ_2	same

	case	rule	θ
S_5	(c)	$\rightarrow^\bullet, \Phi_2 = \emptyset$	$p \rightarrow r$
S_4	(c)	$\vee^\circ$ on F	$p \rightarrow r$
S'_4	(c)	—	$q \rightarrow r$
S_3	(c)	$\vee^\bullet$ in Φ_1	$(p \rightarrow r) \wedge (q \rightarrow r)$
S_2	(c)	$\wedge^\bullet$ in Φ_1	same
S_1	(b)	$\rightarrow^\circ$	same
S_0	(b)	$\wedge^\bullet$ in Φ_2	same

Thus the interpolant constructed is $(p \rightarrow r) \wedge (q \rightarrow r)$

Lemma (CIP for sequents)

Let $\mathcal{D}$ be a focused proof of $\langle - \parallel A^\bullet, B^\circ \rangle$ and let Σ occur in $\mathcal{D}$. Then there exists a formula θ such that

(C1) $\mathcal{I}_L(\Sigma, \theta)$ and $\mathcal{I}_R(\Sigma, \theta)$ are provable;

(C2) $\text{Var}(\theta) \subseteq V_L(\Sigma) \cap V_R(\Sigma)$;

(C3) if Σ is in case (b), then for every α with $\langle - \parallel \Phi_1, \alpha^\circ \rangle$ provable there is α' such that $\langle - \parallel \Phi_1, \alpha'^\circ \rangle$, $\langle - \parallel \alpha'^\bullet, \alpha^\circ \rangle$ and $\langle - \parallel \Psi, \langle \alpha'^\bullet \parallel \Phi_2, \theta^\circ \rangle \rangle$ are provable, with $\text{Var}(\alpha') \subseteq \text{Var}(\alpha) \cup (\text{Var}(\Phi_1) \cap \text{Var}(\Psi))$.

Takeaways

- **One engine.**

binary nested sequent $\Rightarrow$ complete cut-free calculus $\Rightarrow$ reachability $\Rightarrow$ focused proof

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- **CIP.** Induction on the height, but only through unique split

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► **Further topics...**

- Conjecture: Lyndon interpolation, decidability, complexity...
- Future work: extensions, multi-conclusion calculus, translation

Thanks for your attention!

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